



## Induced Fermion Number in the $O(3)$ Nonlinear $\sigma$ Model

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### Abstract

We study the fermion number induced by nontrivial topological configurations in the  $O(3)$  nonlinear  $\sigma$  model in 2+1 dimensions. We consider a scalar background configuration that adiabatically evolves from the normal vacuum to a soliton of winding number unity. The appearance of zero energy modes is analysed as a function of the relative magnitudes of the fermion mass scale,  $m_f$ , and the inverse of the soliton width,  $1/\rho_s$ . We find a single energy level crossing whenever  $m_f > 1/\rho_s$ , implying that, as in the case of the  $O(4)$   $\sigma$  model in four dimensions, the soliton carries the fermion number of any sufficiently heavy fermion. We also show the existence of states with fractional fermion number,  $Q_{GS} = \pm 1/2$ .

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# 1 Introduction

Recently (2+1)-dimensional field theories have received renewed attention as models of systems of statistical mechanics [1]. In particular, the  $O(3)$  nonlinear  $\sigma$  model has been used to describe the long wavelength limit of spin waves of two dimensional quantum antiferromagnets [2] and in modelling superfluid  $^3\text{He}$  films [3], [4]. More recently, the nonlinear  $\sigma$  model coupled to fermions has also been widely studied as a field theoretic realization of a system of interacting spin waves and holons, used to describe the behaviour of high  $T_c$  superconducting materials [5], [6]. While the relevance of this model to high  $T_c$  superconductivity is still under study, the nonlinear  $\sigma$  model by itself has interesting physical properties. The model is defined by the action

$$S_0(\phi) = \frac{1}{2} \int d^3x \partial_\mu \phi_a \partial^\mu \phi_a \quad (1)$$

with  $\phi_a$  a triplet of scalar fields  $(\vec{\phi}, \phi_3)$ . This is an example of a nonlinear classical field theory with topologically nontrivial solutions [7]. Topological solitons occur whenever the space of time independent, finite energy, field configurations is divided into disconnected sectors separated from each other by infinite energy barriers. Such energy barriers can result, for example, from an infinite energy density within a finite region in space. In particular, in the nonlinear  $\sigma$  model, any region in space where  $\phi^2 \neq v^2$  has an infinite energy density. Hence, a finite energy field configuration is a map from the coordinate space into the space of field configurations satisfying the constraint  $\phi^2 = v^2$ , which is  $S^2$ . Compactifying the coordinate space  $R^2$  to  $S^2$  by requiring that the fields approach a unique value at spatial infinity, each finite energy configuration may be characterized by its winding number, i.e., the number of times that the configuration  $\phi(x)$  wraps around the space of finite energy configurations  $S^2$ . Since  $\pi_2(S_2) = Z$ , this implies the existence of solitons of different winding numbers. The

expression for the winding number (topological charge) is

$$n(= Q_{top.}) = \frac{1}{8\pi v^3} \epsilon_{ij} \epsilon_{abc} \int d^2x \phi_a \partial^i \phi_b \partial^j \phi_c. \quad (2)$$

Furthermore, a topological term associated with the nontriviality of the homotopy classes of  $\pi_3(S_2) = Z$  can be included in the action,  $S = S_0 + \theta H$ . The so-called Hopf term

$$H = \pi \int d^3x \epsilon^{\mu\nu\lambda} j_\mu \partial_\nu \frac{1}{\partial^2} j_\lambda, \quad (3)$$

where  $j_\mu$  is the topological current expression leading to eq.(2), breaks P and T invariance. Its appearance may induce nontrivial spin and statistics for the solitons, depending on the parameter  $\theta$ . In our discussion, we do not introduce a Hopf term in the tree-level action. Moreover, the coupling of the scalars to fermions will be explicitly parity invariant. In the absence of parity violating terms in the original Lagrangian, no parity violating Hopf term is induced through radiative corrections[8]-[12].

In even dimensional space-times, the fermionic charge induced by scalar soliton backgrounds has been extensively studied in the literature [13]-[16]. The adiabatic method developed by Goldstone and Wilczek [17] provides a reliable estimate of the induced fermion number for background configurations characterized by small spatial gradients [18]. The essence of the method is to build up the final desired configuration starting from the initial vacuum, by performing slow changes of the fields in space and time. In 2+1 dimensions, in exact analogy with even dimensional systems, the induced fermion current can be evaluated in powers of derivatives of the background field [8], [19]. Keeping the lowest order nonvanishing term in this expansion one obtains the induced charge of the final state in the adiabatic limit,  $Q_{ad.}$ . In this limit of the gradient expansion, the fermion number induced by a soliton background configuration,  $Q_{ind.}$ , is found to be identical to its topological charge,  $Q_{ind.} = Q_{ad.} = Q_{top.}$ <sup>1</sup>. The ground state charge of the system, however, will differ from the induced charge depending on the number of zero energy level crossings  $n_+(n_-)$  in the

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<sup>1</sup>A further analysis in section 3 will support this equality.

positive (negative) direction of the energy axis that occur during the adiabatic evolution,

$$Q_{GS} = Q_{ind.} - (n_+ - n_-) \quad (4)$$

The ground state charge is related to the spectral asymmetry,  $\eta_{[H]}$ , (i.e., the  $\zeta$ -function regularization of the difference between the number of positive and negative eigenvalues of the Dirac Hamiltonian  $H$ ) by the usual expression,  $Q_{GS} = -\eta_{[H]}/2$ <sup>2</sup>. The spectral asymmetry may be written as a sum of its continuous and discontinuous parts,  $\eta_{[H]} = \eta_{[H]}^c + 2(n_+ - n_-)$ . During the adiabatic evolution the induced charge measures the continuous change of the fermion number operator. Its discontinuous changes are those due to the spectral flow contributions, which are overlooked by the adiabatic procedure. [20]-[24].

In this work, we study the fermionic charge induced through vacuum polarization effects by topologically nontrivial scalar configurations in the  $O(3)$  nonlinear  $\sigma$  model in 2+1 dimensions. In particular, we will analyse the conditions under which the ground state fermion charge of the soliton is determined wholly by its topology. In the (3+1)-dimensional  $O(4)$   $\sigma$  model, the appearance of zero energy modes as a function of the relative magnitudes of the fermion's Compton wavelength and the width of the scalar background, has been extensively studied [18], [25]-[27]. It has been shown that a soliton carries the fermion number of any fermion which is sufficiently heavy compared with the typical mass scale of the topological configuration. In the (2+1)-dimensional case, we expect a qualitatively similar result, so that the ground state charge of the system may be identified with the topological charge of the soliton for a fixed range of values for the soliton width, measured in units of the inverse of the fermion mass.

In section 2 we introduce the  $O(3)$  nonlinear  $\sigma$  model coupled to an isodoublet of two component fermions through a Yukawa-type interaction. If the scalar field acquires a vev,  $\langle \phi_3 \rangle = v$ , the fermion acquires mass,  $|m_f| = |vg_y|$ , and  $g_y$ , therefore, defines a characteristic mass scale for the fermion. We will construct the intermediate scalar configuration that will interpolate between the initial and final soliton states. We then write down the energy

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<sup>2</sup>The validity of this expression is restricted to the case when no zero modes of the Hamiltonian appear.

eigenvalue equations for the fermion in a general scalar background configuration, expanding the fermion wave function in a basis of eigenstates of energy and "grand" momentum,  $M_3 = I_3 + j_3$ , the sum of the third components of weak isospin and angular momentum. In section 3, we show that a zero energy mode, if it exists, must appear in the zero "grand" momentum orbital. To that purpose we consider the adiabatic evolution of the background into the limiting case of an infinitely thin final soliton. We then show analytically that at the time of appearance of the zero energy mode the ground state has fractional fermionic charge,  $Q_{GS} = 1/2$ . In section 4, we will use an iterative method to find a numerical solution to the eigenvalue equations for the specific interpolating backgrounds constructed in section 2. Based on the results obtained in the previous section, we need only consider the eigenvalue equations in the  $M_3 = 0$  orbital. We demonstrate the existence of a zero energy mode in the lowest "grand" momentum orbital for different intermediate configurations and obtain the ground state charge of the soliton. Section 5 contains our conclusions.

## 2 Energy Eigenvalue Equations for the Fermion in a Scalar Background

Let us consider the (2+1)-dimensional  $O(3)$  non-linear  $\sigma$  model [7] coupled to an isodoublet,  $\psi$ , of two component fermions, treating the scalar triplet  $\phi_a$  as a background configuration,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + i \bar{\psi} \partial_\mu \gamma^\mu \psi - g_Y \bar{\psi} \phi_a T_a \psi. \quad (5)$$

With the boundary condition at spatial infinity chosen as  $\phi = (0, 0, -1)$ , the soliton with unit winding number has the form

$$\phi = v ( f_2(r) \hat{r}, -f_1(r) ) , \quad (6)$$

where  $\hat{r} = \vec{r}/r$  and  $f_1(r)$  goes monotonically from -1 at  $r = 0$  to 1 at  $r \rightarrow \infty$  and  $f_2(r) \leq 0$  vanishes at  $r \rightarrow 0, \infty$ , but is otherwise negative. We need a parametrization of this expression that allows us to build up the soliton adiabatically from the trivial vacuum state. Notice

that since we are working in the nonlinear limit of the  $\sigma$  model, intermediate configurations with boundary conditions at spatial infinity different from those of the soliton are required in order to interpolate between the initial and final states. Following [27], we choose the interpolating configurations

$$\begin{aligned}\phi_3 &= v \cos[h(t)(\pi - \arccos f_1(r))] , \\ \vec{\phi}(r, t) = (\phi_1, \phi_2) &= v \sin[h(t) \arcsin f_2(r)] \hat{r} \quad r \leq \rho_s , \\ &= -v \sin[h(t)(\pi + \arcsin f_2(r))] \hat{r} \quad r \geq \rho_s ,\end{aligned}\tag{7}$$

where  $\rho_s$  is the soliton width, with the explicit expression for the radial functions,

$$\begin{aligned}f_1(r) &= 1 - 2 \exp[-r^2 \ln(2)/\rho_s^2] \\ f_2(r) &= -2\sqrt{1 - \exp[-r^2 \ln(2)/\rho_s^2]} \times \exp[-r^2 \ln(2)/2\rho_s^2] ,\end{aligned}\tag{8}$$

$h(t)$  being a function which varies slowly and monotonically from 0 to 1 and  $\arcsin f_2$  and  $\arccos f_1$  taking values in the intervals  $[-\pi/2, \pi/2]$  and  $[0, \pi]$ , respectively. The configuration (7) gives a soliton of winding number unity at  $h(t) = 1$ .

The expression for the adiabatic current has been calculated by various authors [19] [8] using the gradient expansion,

$$< j^\mu(x) > = \frac{g_y}{|g_y|} \frac{1}{8\pi|\phi|^3} \epsilon_{\mu\nu\lambda} \epsilon_{abc} \phi_a \partial^\nu \phi_b \partial^\lambda \phi_c ,\tag{9}$$

In the adiabatic limit of the gradient expansion, for positive  $g_y$ , the fermion number induced by a soliton background is found to be identical to the topological charge of the soliton. For  $g_y < 0$ , the sign of the Yukawa coupling changes and so does the fermionic charge. In the following we will assume a positive value of  $g_y$ .

The expression for the fermionic current, eq.(9), becomes ill-defined whenever the scalar field configuration vanishes. In the linear limit of the  $\sigma$  model the change in topological charge could also have been achieved through an intermediate configuration which allows no

fermion flux at spatial infinity (i.e. the scalar field approaches a unique value at spatial infinity during the evolving procedure), but vanishes at some point in space-time. In the context of this work, the nonlinear constraint,  $|\phi| = v$ , precludes such intermediate configurations. No singularity in the fermionic current expression will appear for the interpolating background configuration given by eq.(7). Therefore the adiabatic current, eq.(9), is a reliable expression for the induced fermionic current in this background and  $Q_{ind} = Q_{ad}$  [24], [27], [28]. Moreover, from eqs. (2) and (9), we have  $Q_{ad.} = Q_{top.}$ , so that the induced fermionic charge for the final soliton configuration, eq. (7), reads  $Q_{ind.} = 1$ . To compute the ground state charge of the soliton we must analyse the existence of zero energy level crossings.

In order to look for zero modes in the energy spectrum of the fermion it is useful to write down the Dirac equation in the background of a general scalar triplet

$$\imath \partial^\mu \gamma_\mu \psi - g_y \phi_a T_a \psi = 0. \quad (10)$$

Our conventions for the  $\gamma$  matrices in 2+1 dimensions are:  $\gamma^0 = S_3$  and  $\gamma^i = \imath S^i$ ,  $i=1,2$ , obeying the Clifford algebra  $[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}$  and  $\gamma^\mu \gamma^\nu = g^{\mu\nu} - \imath \epsilon^{\mu\nu\lambda} \gamma_\lambda$ .  $S_a$  and  $T_a$ ,  $a=i,3$ , are the Pauli matrices  $\sigma_a$  in the Dirac and weak isospin spaces, respectively. Eq. (10) becomes,

$$E\psi = -S_i \partial_i (S_3 \psi) + \varphi_a T_a (S_3 \psi) \quad (11)$$

Here, we have defined new variables in terms of the fermion mass scale,  $m_f = v g_y$ , given by  $E \rightarrow E/m_f$ ,  $\vec{x} = \vec{r} m_f$  and  $\rho = \rho_s m_f$ . We have also rescaled the fields,  $\varphi_3 = \phi_3/v$ , and  $\varphi_i = \phi_i/v$ .

Since  $\varphi_3 = \varphi_3(r)$  and  $\varphi_i = \varphi(r) \hat{r}_i$  we observe that the "grand" momentum operator, defined by  $M_3 = j_3 + I_3$ , where  $j_3 = -\imath \partial_\theta + \frac{S_3}{2}$  is the ordinary angular momentum and  $I_3 = \frac{T_3}{2}$  is the isospin, commutes with the Hamiltonian, Eq. (11). Therefore, we look for solutions that are simultaneous eigenstates of energy and "grand" momentum, of the form

$$\psi_{(m)} = \exp(\imath \theta m) \begin{pmatrix} \begin{bmatrix} g_1(x) \exp(-\imath \theta) \\ g_2(x) \end{bmatrix} \\ \begin{bmatrix} g_3(x) \\ g_4(x) \exp(\imath \theta) \end{bmatrix} \end{pmatrix} \quad (12)$$

where  $M_3\psi_{(m)} = m\psi_{(m)}$ ,  $\hat{x} = (\cos\theta, \sin\theta)$  and  $x = |\vec{x}|$ . With this Ansatz for the fermion, eq. (11) reduces to a set of first-order, coupled, differential equations involving the functions  $g_1$ ,  $g_2$ ,  $g_3$  and  $g_4$ :

$$\begin{aligned}\partial_x g_2 &= \frac{m}{x} g_2 - \varphi g_3 + (E - \varphi_3) g_1 \\ \partial_x g_3 &= -\frac{m}{x} g_3 - \varphi g_2 - (E - \varphi_3) g_4 \\ \partial_x g_1 &= -\frac{(1+m)}{x} g_1 - \varphi g_4 - (E + \varphi_3) g_2 \\ \partial_x g_4 &= -\frac{(1-m)}{x} g_4 - \varphi g_1 + (E + \varphi_3) g_3\end{aligned}\tag{13}$$

It is difficult to obtain an analytical solution of eq.(13) except in some limiting cases of the scalar background configuration. For the rest of our discussion in this section, we restrict ourselves to a consideration of the lowest "grand" momentum orbital,  $M_3 = 0$ . This choice will be justified in the next section. Using analytic arguments, we will show there that during the evolution of a narrow soliton of winding number unity a zero-energy mode, if it exists, must appear in the zero "grand" momentum orbital. Moreover, we will show that zero energy modes cannot exist for sufficiently wide solitons.

Therefore, consider the eigenvalue equations, eq.(13), in the case  $m = 0$ . We observe that defining  $g_1 = g_4$  and  $g_2 = -g_3$  in the proposed solution for the fermion field, the set of four coupled equations reduces to

$$\begin{aligned}\partial_x g_2 &= \varphi g_2 + (E - \varphi_3) g_1 \\ \partial_x g_1 &= -\left(\frac{1}{x} + \varphi\right) g_1 - (E + \varphi_3) g_2\end{aligned}\tag{14}$$

Note that there exists another straightforward possibility, which is to define  $g_1^* \equiv g_1 = -g_4$  and  $g_2^* \equiv g_2 = g_3$ . In this case, the coupled equations reduce to

$$\begin{aligned}\partial_x g_2^* &= -\varphi g_2^* + (E - \varphi_3) g_1^* \\ \partial_x g_1^* &= -\left(\frac{1}{x} - \varphi\right) g_1^* - (E + \varphi_3) g_2^*\end{aligned}\tag{15}$$

However, these equations can be obtained from the previous set through the replacement  $E \rightarrow -E$ ,  $\varphi_3 \rightarrow -\varphi_3$ ,  $\varphi \rightarrow -\varphi$  and identifying  $(g_1, g_2)$  in Eq. (14) with  $(-g_1^*, g_2^*)$  in Eq. (15). Recall that in 2+1 dimensions a simultaneous change in the signs of  $\varphi_3$  and  $\varphi$  yields a change in sign of the topological charge. This implies that if a solution exists, for one of the sets of coupled equations, in the background of a soliton with winding number plus one, then a solution with opposite energy will exist for the other set, for a background soliton of topological charge minus one.

Thus, in the zero grand momentum orbital the initial set of four coupled equations reduces to a pair of two coupled equations, which will be solved numerically in section 4.

### 3 Zero Energy Modes in an Infinitely Narrow Soliton Background

We digress to argue, that if a zero energy mode exists, it must appear in the lowest "grand" momentum orbital, with  $m = 0$ . The argument is similar in spirit to that given for the  $O(4)$  model in four dimensions in ref. [26]. From (10), a normalizable zero energy mode is a solution to the equation

$$i\gamma^i \partial_i \psi = g_y \phi_a T^a \psi. \quad (16)$$

In order to show the absence of zero energy modes for sufficiently wide solitons, it is convenient to define the field  $U(x)$ , which belongs to the fundamental representation of  $SU(2)$ , and such that  $\phi_a T^a = UT^3U^\dagger$ . Once the field  $U$  is rotated away, eq.(16) reads

$$i\gamma^i (\partial_i - iA_i) \chi - g_y T^3 \chi = 0 \quad (17)$$

where  $A_i = iU^\dagger \partial_i U$  and  $\chi = U^\dagger \psi$ . The characteristic energy scale of the gauge field  $A_i$  is given by  $1/\rho_s$ . Hence, the perturbations of the free Dirac spectrum will be characterized by  $1/\rho_s$ , and no solution to eq.(17) will exist whenever  $g_y \gg 1/\rho_s$ .

Multiplying eq.(16) by  $T^3$ , and using the relation  $T^3 T^a = \delta_{3a} + \epsilon^{3ab} T^b$ , we get

$$i\gamma^i \partial_i (T^3 \psi) = i g_y \phi_a \epsilon^{3ab} T^b \psi + g_y \phi_3 \psi \quad (18)$$

Multiplying (18) on the left by  $\psi^\dagger$ , and its adjoint on the right by  $\psi$ , and adding the equations gives

$$i\partial_i (\psi^\dagger \gamma^i T^3 \psi) = g_y \psi^\dagger \phi_3 \psi \quad (19)$$

Integrating eq.(19) in space, and assuming the normalizability of the zero energy mode, we get the condition

$$\int d^2 x \psi^\dagger \psi \phi_3(r, t) = 0 \quad (20)$$

From the analysis presented above, it follows that there exists a critical value of the soliton width above which zero energy modes cannot exist. Furthermore, as long as the soliton width is much smaller than  $1/g_y$ , for establishing the existence of a zero energy level crossing the *exact* value of  $\rho_s$  is not crucial. Since soliton configurations of small width differ from each other only in a small volume, this small difference should not affect the existence of solutions [26]. For simplicity, we consider the limiting case  $\rho_s = 0$ . The  $h(t) = 1$  soliton background is then equivalent to the vacuum configuration and, consequently, its ground state fermion number vanishes. The induced fermion number, as given by the adiabatic method, is equal to the soliton winding number. One zero energy level crossing must appear in order to get consistency between the induced and the ground state fermion numbers. From eq.(20) it is apparent that the zero mode must appear at the time  $t_0$  for which  $\phi_3(r, t_0) = 0$ . But eq.(7) tells us that  $\phi_3(r, t)|_{\rho_s=0} = v \cos[h(t)\pi]$ , so that  $h(t_0) = 1/2$  and  $\phi(r, t_0)|_{\rho_s=0} = -v$ .

Consider the equations (13) in the background of the interpolating scalar field from the vacuum to the step soliton final state, at the time  $t_0$ . For  $\rho_s = 0$ ,  $t = t_0$  and  $E = 0$ , eqs.(13) decouple into a pair of coupled equations

$$\begin{aligned} \partial_x g_2 &= m \frac{g_2}{x} + g_3 \\ \partial_x g_3 &= -m \frac{g_3}{x} + g_2 \end{aligned} \quad (21)$$

and

$$\begin{aligned}\partial_x g_1 &= -\frac{g_1}{x} - m\frac{g_1}{x} + g_4 \\ \partial_x g_4 &= -\frac{g_4}{x} + m\frac{g_4}{x} + g_1\end{aligned}\tag{22}$$

Eqs. (21) yield the following second order differential equations for  $g_2(x)$  and  $g_3(x)$ :

$$\begin{aligned}\partial_x^2 g_2 &= g_2 \left[ 1 + \frac{m(m-1)}{x^2} \right] \\ \partial_x^2 g_3 &= g_3 \left[ 1 + \frac{m(m+1)}{x^2} \right]\end{aligned}\tag{23}$$

Analogously, decoupling the first order equations in eq.(22), we obtain

$$\begin{aligned}\partial_x^2 g_4 &= -\frac{2}{x}\partial_x g_4 + g_4 \left[ 1 + \frac{m(m-1)}{x^2} \right] \\ \partial_x^2 g_1 &= -\frac{2}{x}\partial_x g_1 + g_1 \left[ 1 + \frac{m(m+1)}{x^2} \right]\end{aligned}\tag{24}$$

The solutions to these differential equations are well-known. They are modified Bessel functions of fractional order[29], and we write them down for completeness. Explicitly,

$$\begin{aligned}g_2(x) &= A_2 x^{1/2} \mathcal{C}_{\nu_2}(\imath x) & , & \quad \nu_2 = \pm \left| m - \frac{1}{2} \right|, \\ g_3(x) &= A_3 x^{1/2} \mathcal{C}_{\nu_3}(\imath x) & , & \quad \nu_3 = \pm \left| m + \frac{1}{2} \right|, \\ g_1(x) &= A_1 x^{-1/2} \mathcal{C}_{\nu_1}(\imath x) & , & \quad \nu_1 = \pm \left| m + \frac{1}{2} \right| = \nu_3, \\ g_4(x) &= A_4 x^{-1/2} \mathcal{C}_{\nu_4}(\imath x) & , & \quad \nu_4 = \pm \left| m - \frac{1}{2} \right| = \nu_2,\end{aligned}\tag{25}$$

where the  $A_i$  are normalization constants, and the  $\mathcal{C}(x)_{\nu_i}$  are the cylinder functions of order  $\nu_i$ , denoting any one of the Bessel functions of first, second or third kind,  $\mathcal{J}(x)$ ,  $Y(x)$ ,  $H^1(x)$  or  $H^2(x)$ , respectively, or, any linear combination of these. (Recall that the Bessel functions of pure imaginary argument are expressed in terms of the modified Bessel functions  $K_{\nu_i}(x)$  and  $I_{\nu_i}(x)$ , of real argument  $x$ .)

Now, a normalizable solution must satisfy the bound,  $|g_i(x)| < 1/x$ , both when  $x \rightarrow 0$ , and when  $x \rightarrow \infty$ . For half integer orders,  $I_{\nu_i}(x)$  grows exponentially as  $x \rightarrow \infty$  and hence, cannot give a normalizable solution. The functions  $K_{\nu_i}(x)$  are well-behaved at spatial infinity. However,  $x^{-1/2}K_{\nu_i}(x) \geq 1/x$  near the origin for all half integer orders. Consequently no normalizable solutions exist for  $g_1(x)$  and  $g_4(x)$ . For  $g_2(x)$  and  $g_3(x)$ , there is a unique solution well behaved at both limits. It is given by the function  $x^{1/2}K_{1/2}(x)$  (notice that  $K_{1/2}(x) = K_{-1/2}(x)$ ), and corresponds to zero "grand" angular momentum,  $M_3 = 0$ .

In conclusion, for an infinitely narrow soliton only one normalizable solution exists. It has the properties:

$$\begin{aligned} g_1(x) &= g_4(x) = 0, & M_3 &= 0 \\ g_2(x) &= g_3(x) = Ax^{1/2}K_{1/2}(x) . \end{aligned} \quad (26)$$

One can readily prove that at the time  $t_0$  and in the scalar background with  $\rho_s = 0$ , there is a symmetry in the Hamiltonian  $H_{t_0}$  for the zero grand momentum orbital, which gives a one to one correspondence between states of positive and negative energy. Explicitly, one finds that for each solution  $\psi^T = ([g_1 \exp(-i\theta), g_2], [g_3, g_4 \exp(i\theta)])$  of energy  $E$ , there is a solution  $\psi^T = ([-g_1 \exp(-i\theta), g_2], [g_3, -g_4 \exp(i\theta)])$  of energy  $-E$ . The spectral asymmetry therefore vanishes,  $\eta_{[H_{t_0}]} = 0$ . However, since we have proven the existence of a zero energy mode in this background, the ground state charge at time  $t_0$  does not vanish. The correct expression for the ground state charge, after taking into account its degeneracy due to the existence of zero modes of the Hamiltonian  $H_{t_0}$ , reads

$$Q_{GS} = -\frac{\eta_{[H_{t_0}]}}{2} - \frac{1}{2} \left( N_{t_0 B=0}^{emp.} - N_{t_0 B=0}^{occ.} \right) , \quad (27)$$

with  $N_{t_0 B=0}^{emp. (occ.)}$  being the number of zero energy modes at time  $t_0$  that are empty or occupied, respectively. Therefore, the ground state charge of the scalar field at  $t = t_0$  and for  $\rho_s = 0$  is  $Q_{GS} = \pm 1/2$ .

These results are consistent with the predictions of the adiabatic method. In fact, at  $t = t_0$ ,  $Q_{ad.} = Q_{ind.} = 1/2$ . The value  $Q_{GS} = 1/2$  can be obtained by considering the zero

energy mode to be occupied, in which case it must be counted as part of the system's ground state and the induced charge is, consequently, equal to the ground state charge. If the zero energy mode is empty, instead, the ground state charge differs by one unit from the induced charge. Thus,  $Q_{GS} = -1/2$ .

## 4 Ground State Charge of the Soliton

We now use a numerical method to identify fermionic zero energy level crossings in the scalar background configuration given by eq.(7). In order to solve the eigenvalue equations, eqs. (14)-(15), we use a variable order, variable step, Adams technique [30] and the iterative method proposed in reference [31]. We compute the energy eigenvalue of the  $m = 0$  orbital, as a function of the soliton width  $\rho$ , as the scalar field evolves adiabatically from the vacuum to the soliton.

In Fig.1, we plot the energy eigenvalue of eqs. (14) as a function of  $h(t)$ , for fixed values of  $\rho$ . There is a critical value,  $\rho = \rho_c$ , above which no zero energy level crossings develop. This critical value,  $\rho_c = 1$ , is the value of  $\rho$  for which a zero energy mode exists in the final state,  $h(t) = 1$ . We have also computed the energy eigenvalues for eq.(15), where no zero energy level crossing has been found.

As we have explained in section 2, within the framework of the nonlinear  $\sigma$  model the fermionic charge induced by the soliton background is identical to its topological number. Quite generally, in the presence of spectral flow, the ground state charge is instead given by eq. (4). From Fig.1, we see that no level crossings occur for  $\rho > \rho_c$ , while a single zero energy level crossing in the positive direction of the energy axis occurs before the evolution into the final state,  $h(t) = 1$ , for any  $\rho < \rho_c$ . Since the induced charge of the soliton is  $Q_{ind} = n = 1$ , using eq. (4) we obtain

$$\begin{aligned} Q_{GS} &= Q_{ind} - 1 = 0 & \text{for } \rho < \rho_c \\ Q_{GS} &= Q_{ind} = 1 & \text{for } \rho > \rho_c. \end{aligned} \tag{28}$$

We conclude that the fermionic charge is identical to the topological charge for any  $\rho = \rho_* m_f > \rho_c$ . In other words, whenever  $m_f > 1/\rho_*$ .

It is instructive to plot the fermion energy as a function of  $\rho$ , for fixed  $h(t)$ , as was done in Fig. 2. Let  $\rho_{E=0}(t)$  be the value of  $\rho$  for which a zero energy level crossing occurs for some particular  $h(t)$ , thus  $\rho_{E=0}(h(t) = 1) = \rho_c = 1$ . We observe that for any value of  $h(t)$  the  $m = 0$  orbital has positive energy whenever  $\rho < \rho_{E=0}(t)$ . Therefore, in the ground state these levels must be empty. On the contrary, for  $\rho > \rho_{E=0}(t)$  these levels have negative energy and the  $m = 0$  orbital must be filled in the ground state. Since each curve in Fig. 2 is at a fixed  $h(t)$ , there is a fixed induced charge for each curve. Let us denote this fixed charge as  $Q_{ind.} = \beta(t)$ , with  $\beta(t)$  varying from 0 to 1, as  $h(t)$  varies in the same way. From Fig. 1 we know that for  $\rho > \rho_{E=0}(t)$  the ground state charge coincides with the induced charge for any interpolating background. Since the fermion number is decreased by one unit whenever a single level crossing in the positive direction of the energy axis occurs, we see that  $Q_{GS} = \beta(t) - 1$  whenever  $\rho < \rho_{E=0}(t)$ .

As in the previous section, at  $\rho = \rho_{E=0}(t)$ , the ground state is degenerate. If the zero energy mode is occupied, and is therefore counted as part of the system's ground state, the ground state charge is equal to the induced charge,  $Q_{GS} = \beta(t)$ . If, instead, it is empty, the ground state charge will be  $\beta(t) - 1$ . Observe that these two values coincide with the two different values of the ground state charge obtained when taking the limit  $\rho \rightarrow \rho_{E=0}(t)$ , for values of  $\rho$  greater than or lower than  $\rho_{E=0}(t)$ , respectively. In particular, the ground state charge of the final soliton at  $\rho = \rho_c$  may be 0 or 1.

## 5 Conclusions

In this work, we studied the vacuum polarizations effects induced by nontrivial topological configurations in the  $O(3)$  nonlinear  $\sigma$  model in 2+1 dimensions. In particular, we analyse the conditions under which the ground state fermion number of the soliton is given by the

winding number of the topological configuration. We demonstrated that when the soliton is adiabatically constructed from the vacuum configuration, zero energy modes can appear in the fermionic energy spectrum. The existence of zero energy level crossings depends on the relative magnitudes of the fermion mass scale,  $m_f$ , and the inverse of the soliton width,  $1/\rho_s$ . We found a single energy level crossing for a narrow soliton of winding number unity, but no spectral flow contributions appear for a wide soliton. The above implies that the soliton carries the fermion number of any sufficiently heavy fermion.

An additional result of our analysis is the existence of states with fractional fermion number,  $Q_{GS} = \pm 1/2$ . This was proved through the cancellation of the fermionic spectral asymmetry and the existence of a single zero energy fermion mode in a given scalar background. Then, we showed that these values are consistent with the ones obtained by formal considerations based on the adiabatic technique.

It would be interesting to study the consequences of including an explicit odd parity mass term for the fermions. In a future work, we intend to analyse the relationship of the ground state fermion charge and the topological charge of the soliton for this case.

## ACKNOWLEDGEMENTS:

M.C. and C.W. wish to thank W. A. Bardeen and R. D. Peccei for the hospitality extended to them while visiting Fermilab and U.C.L.A., respectively, where portions of this work were done. We would also like to thank E. D'Hoker for useful discussions, and Z. Hlousek and D. Senechal for communicating their results to us.

## FIGURE CAPTIONS

Figure 1: Fermion energy for  $M_3 = 0$  as a function of the evolving backgrounds, for different values of the size parameter  $\rho = m_f \rho_s$ . The full line corresponds to the critical value  $\rho = 1 \simeq \rho_c$  for which a zero mode exists at  $h(t) = 1$ .

Figure 2 : Fermion energy for  $M_3 = 0$  as a function of the size parameter  $\rho$ . Different curves correspond to different scalar backgrounds labelled by the evolution parameter,  $h(t)$ . The energy eigenvalues for a soliton background of winding number unity ( $h(t) = 1$ ) are plotted as a full line.

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Figure 1

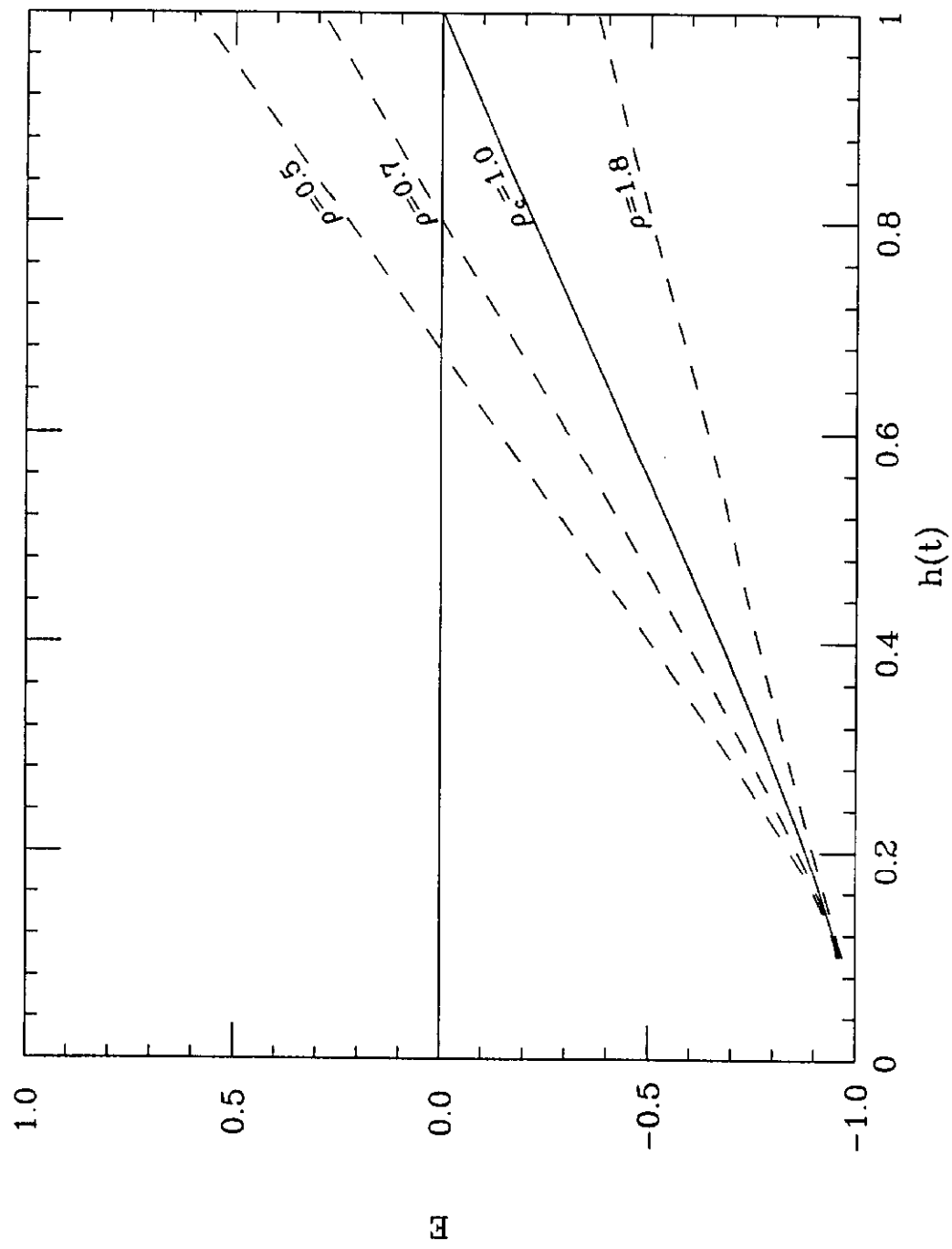


Figure 2

